

An extension of the classical derivative

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1 Introduction

Undergraduate students attending Calculus I in their first semester at college (or increasingly in high school [1]) will learn how to compute the derivative of a function of a real variable, by means of the formula

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}. \quad (1)$$

Some of them may be surprised by the fact that while the derivative of functions like x or x^2 is defined at $x = 0$, a well behaved function like \sqrt{x} is not differentiable at $x = 0$.

Advancing to Calculus II, they will then learn how to find a Taylor or Maclaurin series to represent an elementary function. Once again, they will find that everything seems to break down when they try to expand functions such as $\sin(\sqrt{x})$ at $x = 0$. Of course, a (bad!) way of getting around this is to compute the Maclaurin series of $\sin(x)$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1},$$

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and then replace x by \sqrt{x} , to obtain

$$\sin(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{n+\frac{1}{2}}, \quad x \geq 0. \quad (2)$$

Inquisitive students may ask themselves, What is wrong with fractional powers?

A few of them will finally understand when they take a course in Complex Analysis and learn about branch points, Riemann surfaces, etc. Unfortunately, those will constitute a very small percentage of the whole population of Calculus students.

Over the years, there have been many generalizations of the classical derivative [2]. Most of them are beyond the reach of students at the Calculus I level. In this paper, we propose an extension of the classical derivative that would be easy to teach to Calculus I students and which provides a simple way of computing the expansion of functions at branch points.

In the remainder of this paper, we will always refer to functions whose derivative defined by (1) exists, as differentiable functions.

2 Definition and examples

Definition 1 Let $I \subset \mathbb{R}$ be an interval, $f : I \rightarrow \mathbb{R}$ and $c \in I$. Let $\alpha > 0$ and ω be a real number. We say that a number $D_{\alpha,\omega}[f](c)$ is the (α, ω) -**derivative** of f at c if the limit

$$D_{\alpha,\omega}[f](c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{(x - \omega)^\alpha - (c - \omega)^\alpha} \quad (3)$$

exists. In this case, we say that f is (α, ω) -**differentiable** at c .

We could also define $D_{\alpha,\omega}[f](c)$ by the equivalent limit

$$D_{\alpha,\omega}[f](c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{(c+h-\omega)^\alpha - (c-\omega)^\alpha}.$$

We will use the notation $D_{\alpha,\omega}[f]$ to denote the function whose domain is a subset of I and whose value at c is $D_{\alpha,\omega}[f](c)$. Higher derivatives will be denoted by $D_{\alpha,\omega}^k[f](c)$ and defined by $D_{\alpha,\omega}^0[f](c) = f(c)$ and

$$D_{\alpha,\omega}^{k+1}[f](c) = D_{\alpha,\omega}[D_{\alpha,\omega}^k[f]](c), \quad k = 1, 2, \dots$$

Remark 2 It is clear from the definition (3) that if f is differentiable at $x = c$, then

$$D_{1,\omega} [f] (c) = \frac{df}{dx} (c)$$

for all ω .

Example 3 1. The derivative of the function $f(x) = (x - \omega)^\alpha$ at $x = \omega$ is undefined for all $0 < \alpha < 1$. Could (α, ω) -derivatives do better? Since

$$\frac{f(x) - f(c)}{(x - \omega)^\alpha - (c - \omega)^\alpha} = 1,$$

we conclude from (3) that

$$D_{\alpha,\omega} [(x - \omega)^\alpha] (c) = 1. \quad (4)$$

This shows that the function $f(x) = (x - \omega)^\alpha$ plays the same fundamental role as $f(x) = x$ does for the classical derivative. In particular, we have

$$D_{\frac{1}{2},0} [\sqrt{x}] = 1.$$

2. At $x = 0$, none of the derivatives of the function $f(x) = \sin(\sqrt{x})$ exist. On the other hand,

$$D_{\frac{1}{2},0} [f] (0) = \lim_{x \rightarrow 0} \frac{\sin(\sqrt{x})}{\sqrt{x}} = 1.$$

We will prove in the next section that

$$D_{\frac{1}{2},0}^k [f] (0) = \begin{cases} 0, & k \text{ even} \\ (-1)^{\frac{k-1}{2}}, & k \text{ odd} \end{cases}, \quad k = 1, 2, \dots, \quad (5)$$

and therefore we can write

$$\sin(\sqrt{x}) = \sum_{n=0}^{\infty} D_{\frac{1}{2},0}^n [f] (0) \frac{x^{\frac{n}{2}}}{n!}, \quad x \geq 0$$

rather than (2).

3 Properties

In this section we show some of the basic properties of the (α, ω) -derivative. We start with a theorem that generalizes the usual proof that continuity is a necessary condition for the existence of a derivative and the proofs of the basic differential rules.

Theorem 4 *Let $I \subset \mathbb{R}$ be an interval, $f, g, \rho : I \rightarrow \mathbb{R}$ and $c \in I$. Let $\mathfrak{D}[f](c)$ be defined by*

$$\mathfrak{D}[f](c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{\rho(x)},$$

where $\rho(x)$ is continuous at $x = c$, and $\rho(c) = 0$. Suppose that $\mathfrak{D}[f](c)$ and $\mathfrak{D}[g](c)$ exist. Then,

(i) *f is continuous at $x = c$.*

(ii) *\mathfrak{D} is linear, i.e.,*

$$\mathfrak{D}[af + bg](c) = a\mathfrak{D}[f](c) + b\mathfrak{D}[g](c).$$

(iii) *Product rule:*

$$\mathfrak{D}[f \times g](c) = \mathfrak{D}[f](c) \times g(c) + f(c) \times \mathfrak{D}[g](c).$$

(iv) *Quotient rule:*

$$\mathfrak{D}\left[\frac{f}{g}\right](c) = \frac{\mathfrak{D}[f](c) \times g(c) - f(c) \times \mathfrak{D}[g](c)}{[g(c)]^2},$$

as long as $g(c) \neq 0$.

Proof.

(i) Since

$$f(x) = f(c) + \frac{f(x) - f(c)}{\rho(x)}\rho(x), \quad x \neq c,$$

we have

$$\lim_{x \rightarrow c} f(x) = f(c) + \mathfrak{D}[f](c) \lim_{x \rightarrow c} \rho(x) = f(c).$$

(ii) By definition,

$$\begin{aligned}\mathfrak{D}[af + bg](c) &= \lim_{x \rightarrow c} \frac{(af + bg)(x) - (af + bg)(c)}{\rho(x)} \\ &= \lim_{x \rightarrow c} a \frac{f(x) - f(c)}{\rho(x)} + b \frac{g(x) - g(c)}{\rho(x)} \\ &= a\mathfrak{D}[f](c) + b\mathfrak{D}[g](c),\end{aligned}$$

by the linearity of the limit operation.

(iii) We have

$$\begin{aligned}\mathfrak{D}[f \times g](c) &= \lim_{x \rightarrow c} \frac{(f \times g)(x) - (f \times g)(c)}{\rho(x)} \\ &= \lim_{x \rightarrow c} \frac{f(x)g(x) - f(c)g(x)}{\rho(x)} + \frac{f(c)g(x) - f(c)g(c)}{\rho(x)} \\ &= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{\rho(x)} g(x) + f(c) \frac{g(x) - g(c)}{\rho(x)}.\end{aligned}$$

From part (i), we know that $\lim_{x \rightarrow c} g(x) = g(c)$. Thus,

$$\begin{aligned}\lim_{x \rightarrow c} \frac{f(x) - f(c)}{\rho(x)} g(x) + f(c) \frac{g(x) - g(c)}{\rho(x)} \\ = \mathfrak{D}[f](c) \times g(c) + f(c) \times \mathfrak{D}[g](c).\end{aligned}$$

(iv) In this case,

$$\begin{aligned}\mathfrak{D}\left[\frac{f}{g}\right](c) &= \lim_{x \rightarrow c} \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(c)}{\rho(x)} = \lim_{x \rightarrow c} \frac{f(x)g(c) - g(x)f(c)}{g(c)g(x)\rho(x)} \\ &= \lim_{x \rightarrow c} \frac{1}{g(c)g(x)} \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - g(x)f(c)}{\rho(x)} \\ &= \lim_{x \rightarrow c} \frac{1}{g(c)g(x)} \left[\frac{f(x) - f(c)}{\rho(x)} g(c) - f(c) \frac{g(x) - g(c)}{\rho(x)} \right] \\ &= \frac{1}{[g(c)]^2} \{ \mathfrak{D}[f](c) \times g(c) - f(c) \times \mathfrak{D}[g](c) \}.\end{aligned}$$

■

Remark 5 *Taking*

$$\rho(x) = (x - \omega)^\alpha - (c - \omega)^\alpha$$

in the previous theorem, we find that all the results proven hold for the (α, ω) -derivative.

We shall now find a relation between (α, ω) -derivatives with different values of α and ω . This will provide us with a useful way of computing (α, ω) -derivatives of differentiable functions.

Theorem 6 (i) *Let $c \neq \omega, \zeta$. Then, f is (α, ω) -differentiable at c if and only if f is (β, ζ) -differentiable at c . In this case,*

$$D_{\alpha, \omega} [f] (c) = \frac{\beta (c - \zeta)^{\beta-1}}{\alpha (c - \omega)^{\alpha-1}} D_{\beta, \zeta} [f] (c).$$

(ii) *If f is (β, ω) -differentiable at ω then,*

$$D_{\alpha, \omega} [f] (\omega) = \begin{cases} 0 & \text{if } \alpha < \beta \\ \text{undefined} & \text{if } \alpha > \beta \end{cases}.$$

Proof.

(i) It follows from (3) that

$$\begin{aligned} D_{\alpha, \omega} [f] (c) &= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{(x - \omega)^\alpha - (c - \omega)^\alpha} \\ &= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{(x - \zeta)^\beta - (c - \zeta)^\beta} \frac{(x - \zeta)^\beta - (c - \zeta)^\beta}{(x - \omega)^\alpha - (c - \omega)^\alpha}. \end{aligned}$$

From L'Hopital's rule,

$$\lim_{x \rightarrow c} \frac{(x - \zeta)^\beta - (c - \zeta)^\beta}{(x - \omega)^\alpha - (c - \omega)^\alpha} = \lim_{x \rightarrow c} \frac{\beta (x - \zeta)^{\beta-1}}{\alpha (x - \omega)^{\alpha-1}},$$

and the result follows immediately.

(ii) We have

$$\begin{aligned} D_{\alpha, \omega} [f] (\omega) &= \lim_{x \rightarrow \omega} \frac{f(x) - f(\omega)}{(x - \omega)^\alpha} = \lim_{x \rightarrow \omega} \frac{f(x) - f(\omega)}{(x - \omega)^\beta} (x - \omega)^{\beta-\alpha} \\ &= D_{\beta, \omega} [f] (\omega) \times 0, \end{aligned}$$

as long as $\alpha < \beta$.

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When $\beta = 1$, we obtain the following result.

Corollary 7 *If f is differentiable at x , then*

$$D_{\alpha,\omega}[f](x) = \frac{1}{\alpha} (x - \omega)^{1-\alpha} f'(x) \quad (6)$$

for all $x \neq \omega$.

Example 8 1. Let $f(x) = (x - a)^p$. Then, we obtain from (6)

$$D_{\alpha,\omega}[f](x) = \frac{p}{\alpha} (x - \omega)^{1-\alpha} (x - a)^{p-1}.$$

If we take $p = \alpha$ and $a = \omega$, we recover (4).

2. Let's apply (6) to the function $f(x) = \sin(\sqrt{x})$. We have

$$D_{\alpha,\omega}[f](x) = \frac{1}{\alpha} (x - \omega)^{1-\alpha} \frac{\cos(\sqrt{x})}{2\sqrt{x}}.$$

In particular,

$$D_{\frac{1}{2},0}[f](x) = \cos(\sqrt{x}),$$

and using (6) again, we get

$$D_{\frac{1}{2},0}^2[f](x) = -\sin(\sqrt{x}).$$

In a similar fashion, we find

$$D_{\frac{1}{2},0}^k[f](x) = \begin{cases} (-1)^{\frac{k}{2}} \sin(\sqrt{x}), & k \text{ even} \\ (-1)^{\frac{k-1}{2}} \cos(\sqrt{x}), & k \text{ odd} \end{cases},$$

which proves (5).

Conclusion 9 *We have extended the usual definition of a derivative in a way that Calculus I students can easily comprehend and which allows calculations at branch points. In a forthcoming paper, we will prove the (α, ω) -versions of the classical theorems (Rolle, Lagrange, Cauchy, L'Hopital, Taylor, etc.) and consider possible extensions to complex variables.*

References

- [1] D. M. Bressoud. The changing face of calculus: First-semester calculus as a high school course. *FOCUS*, 24(6):6–8, 2004.
- [2] A. M. Bruckner and J. L. Leonard. Derivatives. *Amer. Math. Monthly*, 73(4, part II):24–56, 1966.